§12) Holomorphic families of rational maps

Def: Let Λ be a connected complex manifold (possibly singular).

A holomorphic family of rational maps (in $\mathbb{P}^1$), parametrised by Λ, is a holomorphic map $f : \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$
$(\lambda, z) \mapsto f_\lambda(z)$

(Holomorphic = holomorphic in each coordinate.)

Examples: $\Lambda = \mathbb{C}$, $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ family of quadratic polynomials
$c, z \mapsto z^2 + c$ (extends to $f : \mathbb{C} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$)

For cubic poly: $(c,a) \in \mathbb{C}^2$, $f_{ca}(z) = \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3$, $\mathcal{C}(f_{c,a}) = \{0, c\}$ i.e. critical points are marked. *

Rem: $\forall \lambda \in \Lambda$, $f_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map on $\hat{\mathbb{C}}$, of some degree d_λ.

In the previous example $d_\lambda \equiv 2$.

d_λ in general may drop. We call this phenomenon "degeneration".

We will assume $d_\lambda \equiv d \geq 2$ from now on.

We will also assume that the family $(f_\lambda)_{\lambda \in \Lambda}$ has marked critical points:

Def: * the family $f : \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $(f = (f_\lambda)_{\lambda \in \Lambda})$ has marked critical points if there exist maps $\lambda \in \Lambda \mapsto c_{j,\lambda} \in \hat{\mathbb{C}}$, $j = 1 \ldots 2d-2$, so that $\mathcal{C}_{f_\lambda} = \{c_{1,\lambda}, \ldots, c_{2d-2,\lambda}\}$

(In general, one could need multivalued maps to follow critical points).

We will also denote $F_\lambda := F(f_\lambda)$ and $J_\lambda = J(f_\lambda)$

We want to understand how J_λ (F_λ) varies with λ.

Def: If for any j, the maps $\{\lambda \mapsto f_\lambda^n(c_{j,\lambda}), n \in \mathbb{N}\}$ form a normal family on a neighborhood of $\lambda_0 \in \Lambda$, then we say that λ_0 is J-stable.

We denote by $\mathrm{Stab}(f)$ the set of J-stable parameters. Its complementary set $\Lambda \setminus \mathrm{Stab}(f)$
open set

is called the Bifurcation locus $Bif(f)$ (or the J-unstable locus)
$\nwarrow$ closed set

Proposition. Let $(f_c)_{c\in\mathbb{C}}$, $f_c(z) = z^2+c$ be the Mandelbrot family of quadratic polynomials.

Let $\mathcal{M} = \{c\in\mathbb{C} \mid (f_c(0))_{c\in\mathbb{C}}$ is bounded$\}$

Then $Bif(f) = \partial\mathcal{M}$.

Proof: $\forall c_0 \in \mathbb{C}\setminus\mathcal{M}$, and U nbhd of c_0 in $\mathbb{C}\setminus\mathcal{M}$, then $\forall c\in U$, $(f_c^n(0))$ diverges uniformly from compacts of $\mathbb{C}$. $\Rightarrow c_0 \in Stab(f)$

$\forall c_0 \in \mathring{\mathcal{M}}$, and U nbhd of c_0 in $\mathring{\mathcal{M}}$, $(f_c^n(0))_n$ is bounded. In particular $c\mapsto f_c^n(0)$ has images in a ~~...~~ hyperbolic R-S. and is hence normal, and $c_0 \in \mathbb{C}^{stable}$.

$\forall c_0 \in \partial\mathcal{M}$, $\forall U$ nbhd of c_0, $U\cap(\mathbb{C}\setminus\mathcal{M}) \neq \emptyset$, and any limit function cannot be continuous. Hence $c_0 \in Bif(f)$. $\square$

To study how J_λ and F_λ varies, it is natural to study how periodic points vary. In fact, recall that $J_f = \overline{Rep(f_\lambda)}$, while Fatou components are preperiodic associated to periodic cycles (but for Herman rings).

Rem: If z_0 is a fixed point of f_{λ_0} with multiplier $f_{\lambda_0}'(z_0)\neq 1$, then there exists a (unique) holomorphic function $\lambda\mapsto z(\lambda)$ defined on a nbhd of λ_0 and such that $z(\lambda_0) = z_0$, $z(\lambda) = f_\lambda(z(\lambda))$.

This follows from the implicit function theorem applied to $f_\lambda(z) - z = 0$.

We now investigate this ~~phenomenon~~ for non-local perturbations.

Prop: Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of rational maps of constant degree $d\geq 2$, and $U\subset\Lambda$ a simply connected open set.

Let z_0 be a repelling fixed point of f_{λ_0}, $\lambda_0\in U$, and $\lambda\mapsto z(\lambda)$ the map locally defined on a neighborhood of λ_0 as in the Remark.

Then, either:

(i) $\lambda \mapsto z(\lambda)$ prolonges holomorphically on U, and $z(\lambda)$ is repelling for f_λ for all $\lambda \in U$; or.

(ii) $\lambda \mapsto z(\lambda)$ prolonges holomorphically along a path γ from λ_0 to λ_1 in U so that $z(\lambda_1)$ is an attracting fixed point of f_{λ_1}.

Proof: Let $\mathcal{E}_0$ be the connected component of the analytic set $\{(\lambda, z) \in U \times \hat{\mathbb{C}} \mid f_\lambda(z) = z\}$ containing (λ_0, z_0).

If $(\lambda, z) \in \mathcal{E}_0$ and $f_\lambda'(z) \neq 1$, then (λ, z) is a regular point of $\mathcal{E}_0$ (by implicit function theorem).

If $f_\lambda'(z) \neq 1 \;\; \forall (\lambda, z) \in \mathcal{E}_0$, then $\lambda \mapsto z(\lambda)$ prolonges to all U (since U is simply connected).

If $|f_\lambda'(z(\lambda))| > 1 \;\; \forall \lambda \in U$, we are in case (i).

If not, the map $(\lambda, z) \mapsto f_\lambda'(z)$ restricted to $\mathcal{E}_0$ is not constant, hence it is open. Under the assumption that $|f_\lambda'(z(\lambda))|$ is not $> 1 \;\; \forall \lambda \in U$, we conclude that there exists $(\lambda_1, z_1) \in \mathcal{E}_0$ with $|f_{\lambda_1}'(z_1)| < 1$. (this will be case (ii)).

In fact, in this case we can avoid (again by openness) the locus where $f_\lambda'(z) = 1$ (which contains the singular set of $\mathcal{E}_0$), and find a path γ where $\lambda \mapsto z(\lambda)$ prolonges holomorphically and joins λ_0 and λ_1.

$\square$

Holomorphic motions and λ-lemma.

These motions have been introduces by Mañé, Sad, Sullivan.

Def: Let $\lambda_0 \in \Lambda$. A holomorphic motion of a set $E = E_{\lambda_0} \subset \hat{\mathbb{C}}$ parametrised by (Λ, λ_0) is a family of injections $\Phi_\lambda : E \to \hat{\mathbb{C}}$, $\lambda \in \Lambda$, such that $\Phi_{\lambda_0} = \mathrm{Id}_E$ and such that $\lambda \mapsto \Phi_\lambda(z)$ is holomorphic for any $z \in E$

Example: this is the content of the previous proposition with $E_{\lambda_0} = \{z_0\}$, at least in case (c).

Theorem (λ-lemma) A holomorphic motion of a set E admits a unique extension to a holomorphic motion of $\overline{E}$.

Moreover, this extension $\Phi : \Lambda \times \overline{E} \to \hat{\mathbb{C}}$ is continuous, and $\forall \lambda \in \Lambda$, the map $\Phi_\lambda : \overline{E} \to \hat{\mathbb{C}}$ is the restriction of a quasiconformal map of $\hat{\mathbb{C}}$.

We won't see the proof of this result, but some of its applications.

Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of rational functions.

Denote by $E = E_{\lambda_0}$ the set of repelling cycles of f_{λ_0} ($E = Rep(f_{\lambda_0})$).

Since repelling cycles are stable under perturbation, any cycle $Z \subset E$, $Z = \{z_0, \dots, z_{m-1}\}$, follows a holomorphic motion Φ_λ, with λ in a suitable nbhd $U_Z \subset \Lambda$ of λ_0.

<u>If</u> $\exists U$ open nbhd of λ_0, $U \subset \bigcap_{\substack{Z \subset E \\ \text{cycle}}} U_Z$, then $\Phi_\lambda : E \to \hat{\mathbb{C}}$ defines a holomorphic motion. It follows from the λ-lemma and the fact that $\overline{E_{\lambda_0}} = \overline{Rep(f_{\lambda_0})} = J_{\lambda_0}$ that we have a holomorphic motion of J_{λ_0}.

It turns out (mainly by uniqueness of the extension of a holomorphic motion) that the holomorphic motion of J_{λ_0} is compatible with the dynamics, in the following sense:

Definition: Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of rational maps, and $\lambda_0 \in \Lambda$. We say that the Julia sets J_λ vary holomorphically if there exist a holomorphic motion $\Phi_\lambda : J_{\lambda_0} \to \hat{\mathbb{C}}$ s.t. $\Phi_\lambda(J_{\lambda_0}) = J_\lambda$ and such that

$$\Phi_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ \Phi_\lambda(z) \quad \forall z \in J_{\lambda_0}. \qquad (*)$$

Rem: all these concepts can be expressed locally at $\lambda_0 \in \Lambda$ by replacing Λ with a nbhd U of λ_0 in Λ.

Rem $(*)$ comes from the statement for $Rep(f_\lambda)$, plus a continuity argument.

Theorem. Let $(f_\lambda)_{\lambda\in\Lambda}$ a holomorphic family of rational fractions, and $\lambda_0\in\Lambda$. (12.5)

Then λ_0 is J-stable $\Leftrightarrow$ J_{λ_0} varies holomorphically (on a nbhd of λ_0)

Proof ($\Leftarrow$) Suppose that J_{λ_0} varies holomorphically.

Let $z_1, z_2, z_3 \in J_{\lambda_0}$ be three points not in $PC(f_{\lambda_0})$.

From the conjugacy relation $(*)$, we infer that $\Phi_\lambda(z_j)\notin PC(f_\lambda)$.

In fact, we just need to care for $PC(f_\lambda)\cap J_{f_\lambda}$.

The critical points in J_{f_λ} also follow the holomorphic motion, and they send critical points to critical points because $(*)$ is valid on the totally invariant sets J_{f_λ}, and we just count the number of preimages to compute the local degree.

Again by $(*)$ the orbits of the critical points in J_{f_λ} follow the holomorphic motion.

It follows by Montel theorem that $\{\lambda\mapsto f_\lambda^n(c_{\lambda,j})\}$ is a normal family, ie, $\lambda_0\in Stab(f)$

(the three maps $\Phi_\lambda(z_j)$ are all distinct by injectivity of holomorphic movement, an distinct from $\lambda\mapsto f_\lambda^n(c_{\lambda,j})$. We used already this argument: it is Montel plus a composition with the cross ratio between the four points.

($\Rightarrow$) Suppose $\lambda_0\in Stab(f)$, and U a nbhd of λ_0 where $\{\lambda\mapsto f_\lambda^n(c_{\lambda,j})\}$ is normal.

We claim that the local stability of the repelling cycles holds on all U.

This will imply the holomorphic motion of J_λ on U as explained above.

We may assume $z(\lambda_0)$ to be a repelling fixed point (up to replacing f with an iterate)

Suppose by contradiction that $z(\lambda)$ (given by the implicit function theorem) is not defined on the whole U.

By the previous prop, this would imply that we can find a path between λ_0 and $\lambda_1\in U$ so that $|f_{\lambda_1}'(z(\lambda_1))|<1$ (attracting fixed point).

This would contradict the normality of the critical orbits:

- $z(\lambda)$ is attracting in a nbhd V^+ of λ_1,
- We know that $\exists\, c_\lambda$ critical point in the basin of $z(\lambda)$:

$f_\lambda^n(c_\lambda) \xrightarrow[n\to\infty]{} z(\lambda)$.

Hence $\lambda \mapsto f_\lambda^n(c_\lambda)$ converges to the map $\lambda \mapsto z(\lambda)$ on V^+.

Similarly, $z(\lambda)$ is repelling in a nbhd V^- of λ_0, and the orbit of c_λ for $\lambda \in V^-$ does not converge towards $z(\lambda)$.

Hence $\{f_\lambda^n(c_\lambda)\}$ is not normal on U. (the limit of any subsequence should be $z(\lambda)$ on the whole U by analytic continuation, but it cannot). $\square$

Other characterisations of $\mathrm{Stab}(f)$.

Theorem: $(f_\lambda)_{\lambda\in\Lambda}$ holomorphic family of rational maps in $\mathbb{P}^1$, $\lambda_0 \in \Lambda$. TFAE:

(i) $\lambda_0 \in \mathrm{Stab}(f)$

(ii): $\exists U$ nbhd of λ_0, $\forall \lambda \in U$, $c_{i,\lambda} \in J_\lambda \iff c_{i,\lambda_0} \in J_{\lambda_0}$.

(iii) The number $N_a(\lambda)$ of contracting cycles of f_λ is constant in a nbhd of λ_0

(iv) $\exists U$ nbhd of λ_0 s.t. any neutral cycle of f_λ, $\lambda \in U$ is persistent.

Def: A neutral (indifferent) periodic point z_{λ_1} of f_{λ_1} is persistent (indifferent) if:

$\exists\, z : U \to \hat{\mathbb{C}}$, $z(\lambda_1) = z_{\lambda_1}$, $f_\lambda^m(z(\lambda)) = z(\lambda)$ and $|(f_\lambda^m)'(z(\lambda))| = 1$ $\forall \lambda \in U$.

Proof: (i $\Rightarrow$ ii) $\lambda_0 \in \mathrm{Stab}(f) \Rightarrow J_\lambda$ varies holomorphically.

$\leadsto$ the local degree of points $z(\lambda) \in J_\lambda$ is preserved by the holomorphic motion (see previous proof) $\Phi_\lambda^u(z)$, and we conclude.

(ii $\Rightarrow$ iii) The idea is to control the convergence of orbits of critical points towards contracting cycles.

Pick three repelling periodic points $z_1(\lambda_0), z_2(\lambda_0), z_3(\lambda_0)$.

Up to shrinking U, we may assume such points can be followed holomorphically:

$\exists\ \lambda \mapsto z_j(\lambda) \in J_\lambda \quad , \lambda \in U.$

By hypothesis, $c_\lambda \in F_\lambda \Leftrightarrow c_{\lambda_0} \in F_{\lambda_0}$.

If this is the case, the $\lambda \mapsto f^n_\lambda(c_\lambda)$ avoids $z_j(\lambda)$ $j = 1,2,3$, and by Montel (+ cross ratio argument), that $\mathcal{F}$ is normal.

Let $\lambda_1 \in U$ be any parameter, and z_1 an attracting fixed point for f_{λ_1} (we replaced f_λ by some iterate f_λ^m).

Then $\exists\ c_{\lambda_1} \in F_{\lambda_1} (\cap \mathcal{C}_{f_{\lambda_1}})$ so that $f^n_{\lambda_1}(c_{\lambda_1}) \xrightarrow[n]{} z_1$

Let $z(\lambda)$ be the unique holomorphic map defined in the nbhd of λ_1 obtained by following z_1: $z(\lambda)$ is fixed, attracting (if $\|\lambda - \lambda_1\| \ll 1$) and $z(\lambda_1) = z_1$.

By equicontinuity of $\lambda \mapsto f^n_\lambda(c_\lambda)$, we obtain that $f^n_\lambda(c_\lambda)$ converges towards $z(\lambda)$ for $\|\lambda - \lambda_1\| \ll 1$. (in a nbhd U_1 of λ_1)

In particular, all adherence values of $f^n_\lambda(c_\lambda)$ are given by $z(\lambda)$ on this nbhd.

We extend $z(\lambda)$ in all U, by setting $z(\lambda) = \lim_{n\to\infty} f^n_\lambda(c_\lambda)$

(we saw that $\{\lambda \mapsto f^n_\lambda(c_\lambda)\}$ is normal on U, and converges on U_1, hence converges on U).

By construction, $f_\lambda(z(\lambda)) = z(\lambda)$, i.e., $z(\lambda)$ is a fixed point of f_λ, and it must be contracting, since a critical point converges towards it.

Property (iii) follows

(iii $\Rightarrow$ i) Let N_λ be the number of contracting cycles of f_λ.

We suppose $N_\lambda \equiv N_{\lambda_0}$ in a nbhd U of λ_0.

If z_0 is a repelling cycle of f_{λ_0}, then either 1) it stays repelling, or 2) it becomes attracting for some $\lambda \in U$. This second option contradicts the constance of N_λ.

This implies that $\mathrm{Rep}(f_\lambda)$ moves holomorphically and so does J_λ, its closure.

(i $\Rightarrow$ iv). We have that J_λ varies holomorphically on a nbhd U of λ_0.

Let $z(\lambda_1)$ an indifferent fixed point of f_{λ_1}.

- If $z(\lambda_1) \in F_{\lambda_1}$, then it is a Siegel point.

Being the multiplier $\neq 1$, we can follow it holomorphically and get $z(\lambda) = f_\lambda(z(\lambda))$.

If $f'_\lambda(z(\lambda))$ is not constant, then there are values of λ arbitrarily close to λ_1 s.t. ~~z~~ $z(\lambda)$ is repelling $\Rightarrow z(\lambda) \in J_\lambda$, against the existence of a holomorphic motion.

- If $z(\lambda_1) \in J_{\lambda_1}$, then it is the holomorphic motion that gives a fixed point $z(\lambda) := \phi_\lambda(z(\lambda_0))$ in J_λ. In particular $|f'_\lambda(z(\lambda))| \geq 1$. For λ close to λ_1, and $f'_\lambda(z(\lambda))$ is constant on λ (or it would be open, against).

Hence $z(\lambda_1)$ is an persistent indifferent fixed point.

(iv $\Rightarrow$ iii) Let U be a simply connected nbhd of λ_0 where contracting cycles stay contracting (by taking U small), and all neutral cycles stay neutral (by (iv)).

If $\lambda_0 \notin \mathrm{Stab}(f)$, by previous results and λ-lemma, $\exists$ repelling cycle z_0 of f_{λ_0} not stable on U, and by the Proposition of the beginning, a path γ in U

joining λ_0 and λ_1, and a holomorphic (multivalued) map $\lambda \mapsto \gamma(\lambda)$, defined in a nbhd of γ, so that $\gamma(\lambda_0) = \gamma_0$, and γ_{λ_1} is an attracting cycle.

By continuity $\exists\, \lambda_2$ in γ so that $\gamma(\lambda_2)$ is an indifferent cycle.

But the $\gamma(\lambda_2)$ is a non persistent indifferent cycle: a contradiction. $\square$

Corollary: $Stab(f)$ is an open and dense subset of Λ.

Proof: $Stab(f)$ is ~~the~~ open by definition.

Let $\lambda_0 \in \Lambda$ be any parameter. Denote by $N(\lambda) = \#$ ~~attracting~~ contracting cycles of f_λ (rem: $N(\lambda) \leq 2d-2$, $d = \deg f_\lambda$):

Any neighborhood ~~U~~ U of λ_0 contains a parameter λ_U which maximises $N(\lambda)$ on U (just because $N(\lambda) \leq 2d-2$ uniformly).

But contracting cycles are stable under small perturbations, and $N(\lambda)$ can only increase (and not decrease) under small perturbation.

Hence $N(\lambda) \equiv N(\lambda_U)$ on a neighborhood of λ_U; which gives $\lambda_U \in Stab(f)$ from the previous theorem.

Hence $Stab(f)$ is dense in Λ. $\square$

Hyperbolic maps and Stable parameters.

Hyperbolic maps are a class of rational maps $f: \hat{\mathbb{C}} \circlearrowleft$ with "nice" dynamical properties. (see Milnor, §19)

Def: A rational map $f: \hat{\mathbb{C}} \circlearrowleft$ is called (dynamically) hyperbolic if f is expanding on its Julia set $J = J_f$:

there is a conformal metric μ, defined on a neighborhood V of J, so that the derivative df_z at any point $z \in J$ satisfies the inequality

$$\| df_z(v) \|_\mu > \|v\|_\mu \quad \forall v \in T_z \hat{\mathbb{C}},\ v \neq 0.$$

Rem: being J compact, this implies that there exists $k > 1$ so that

$$\| df_z \|_\mu \geq k \quad \forall z \in V$$

Theorem: let f be a rational map $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$. TFAE:

(1) f is (dynamically) hyperbolic.

(2) $\overline{PC(f)} \cap J(f) = \emptyset$.

(3) the orbit of any critical point converges towards a contracting cycle

Rem: In this case, the orbit of any $z \in F(f)$ converges to an attracting cycle. In particular, we have no parabolic cycles. Moreover, since boundary of Siegel disks and cremer points belong to $\overline{PC(f)} \cap J(f)$, we don't have them either, and all cycles of f are either contracting or repelling.

Other properties: 1.) $J(f)$ has area 0. (see Bertolot-Mayer)

2) If $J(f)$ is connected, then it is locally connected (see Milnor)

3) $\dim_H J_f < 2$ where $\dim_H$ is the Hausdorff dimension. see [Bertelot-Mayer] (12.11)

Proof of the Theorem:

Set $U = \hat{\mathbb{C}} \setminus \overline{PC(f)}$ and $V = f^{-1}(U)$

As in the proof of "boundary of Siegel and Cremer in $\overline{PC(f)}$",

$f: V \to U$ is a d-fold covering map (since we avoid critical points/values)

We may also assume $f \neq z \mapsto z^{\pm d}$, and all connected components of U, V are hyperbolic.

$(2 \Rightarrow 1)$: Suppose $\overline{PC(f)} \cap J(f) = \emptyset$, i.e., $J(f) \subset U$. $\Rightarrow F(f) \supset U$.

If $V = U$, then $\{f^n|_U\}$ is normal (by hyperbolicity of V), and we get a contradiction.

More precisely, any connected component of V that intersect J must be smaller than the corresponding connected component of U.

It follows (as in that thm) that $\|df_z\|_V > 1$ $\forall z \in V$.
poincaré metric

Since $J(f) \subset V$, f is (dynamically) hyperbolic.

$(3 \Rightarrow 2)$ see the remark above.

$(1 \Rightarrow 3)$ We have that f is expanding with respect to a conformal metric on some neighborhood W of $J(f)$ (we could actually just work with the spherical metric). In particular, $C(f) \cap W = \emptyset$ (critical points are not expanding ...) Let $N_\varepsilon(J)$ be the ε-neighborhood of J. (w.r.t. μ)

Pick $\varepsilon \ll 1$ so that 1) $\forall p \in N_\varepsilon(J)$ can be joined to J in W by at least one minimal geodesic.

2) $f^{-1}(N_\varepsilon(J)) \subset W$.

For any $z \in f^{-1}(N_\varepsilon(J))$, $\operatorname{dist}(z, J) \leq \operatorname{dist}(f(z), J)/k$ $(*)$

It follows that $\forall z_0 \in F_f$. $\#\{n \in \mathbb{N} \mid f^n(z_0) \in N_\varepsilon(J)\} < +\infty$.

In fact no point of $N_\varepsilon(J)^c$ can map inside $N_\varepsilon(J)$ (it would go against the expansion (*)), while if $z_0 \in N_\varepsilon(J) \setminus J$, then the distance from J increases until it is $> \varepsilon$, and we are outside $N_\varepsilon(J)$.

Then any accumulation point $\hat{z}$ of $(z_n = f^n(z_0))$ must be in F_f.

Let $\hat{U}$ be the c.c. of F_f containing $\hat{z}$. Then $\exists m \geq 1$, $f^m(U) = U$.

U can be only a contracting basin, a parabolic basin or a rotation domain.

But it cannot be parabolic (the unique accumulation point is the fixed point in J_f), nor a rotation domain:

the boundary of rotation domains is in $S(f)$ $\Rightarrow$ we would have invariant circles for the action of f in N_ε, against (*).

This implies that all ~~cont~~ fixed Fatou components are contracting basins. In particular, ~~every~~ the orbit of every critical point converges toward a contracting cycle.

$\square$

We come back to holomorphic families, showing

Theorem: Let $f = (f_\lambda)_{\lambda \in \Lambda}$ be a $\overset{\text{hol.}}{\text{family}}$ of rational maps of degree $d \geq 2$, and H be the set of $\lambda \in \Lambda$ so that f_λ is (dynamically) hyperbolic. Then H is a open and closed subset of $Stab(f)$.

Proof: If f_λ is hyperbolic, then all critical orbits converge towards contracting cycles. This follows that $\{\lambda \mapsto f_\lambda^n(c_{\lambda,j})\}$ are normal families, and $\lambda \in Stab(f)$. Being contracting cycles stable this also implies that H is open.

Let $\lambda_0 \in Stab(f) \setminus H$. Being f_{λ_0} not hyperbolic, $\exists\, c_{\lambda_0} \in J_{\lambda_0}$ or such that $(f^n_{\lambda_0}(c_{\lambda_0}))$ has accumulation to J_{λ_0}.

In the first case: λ_0 admits a neighborhood U such that the critical points c_λ of f_λ belong to J_λ $\forall \lambda \in U$ (by holomorphic motion).

In the second case, f_{λ_0} has a parabolic cycle:

(Fatou components (the one containing c_{λ_0}) is preperiodic, and only parabolic basins have orbits accumulating to J_{λ_0}).

This parabolic cycle is persistent (by stability). ~~In particular~~

In both cases, we obtain that f_λ is still non hyperbolic for λ close to λ_0 in $Stab(f)$, and $Stab(f) \setminus H$ is open. $\square$

This result implies that a connected component U of $Stab(f)$ contains a hyperbolic map $\Leftrightarrow$ f_λ is hyperbolic $\forall \lambda \in U$.

Definition: A connected component U of ~~A~~ $Stab(f)$ is called hyperbolic if $\exists \lambda \in U$ $(\forall \lambda \in U)$, f_λ is a hyperbolic map.

Hyperbolicity conjecture: Every connected component of $Stab(f)$ is hyperbolic ($\Leftrightarrow$ hyperbolic ~~maps~~ parameters are dense in Λ).

This conjecture is open (even for $f_c(z) = z^2 + c$).

Bifurcation locus and pluripotential theory.

We saw that, given $f_\lambda : \hat{\mathbb{C}} \circlearrowleft$ rational map, one can construct its equilibrium measure μ_λ, so that $pr^* \mu_\lambda = $ "$\frac{i}{\pi} \partial\bar\partial G_\lambda$", (in the sense of distribution/currents) with $G_\lambda = \lim_{n\to\infty} \frac{1}{d^n} \log \| F_\lambda^n(z) \|$ the Green function associated to F_λ the lift of f_λ to $\mathbb{C}^2 \setminus \{0\}$.

There is an analogous result (due to De Marco); for families:

Theorem (de Marco): $f = (f_\lambda)_{\lambda\in\Lambda}$ holomorphic family of rational maps on $\mathbb{P}^1$, then there exists a closed positive $(1,1)$-current T_{bif} on Λ so that ~~support~~ $\mathrm{supp}\, T_{bif} = \mathrm{Bif}(f)$.

current := "form with distributions as coefficients".

Idea of construction: Define $H(\lambda) = \sum_j G_\lambda(c_{\lambda,j})$, where $c_{\lambda,j}$ are the marked critical points.

$H(\lambda)$ is a plurisubharmonic function on $\lambda \in \Lambda$, and the current T_{bif} is defined as: $T_{bif} = \frac{i}{\pi} \partial\bar\partial H$.